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The Gauss Law Operator Algebra and Double Commutators in Chiral Gauge Theories

J. M. Pawlowski[1]
Theor.–Phys. Institut, Universität Jena
Fröbelstieg 1
D–07743 Jena
Germany

Abstract

We calculate within an algebraic Bjorken–Johnson–Low (BJL) method anomalous Schwinger terms of fermionic currents and the Gauss law operator in chiral gauge theories. The current algebra is known to violate the Jacobi identity in an iterative computation. Our method takes the subtleties of the equal–time limit into account and leads to an algebra that fulfills the Jacobi identity. The non-iterative terms appearing in the double commutators can be traced back directly to the projective representation of the gauge group.

[1]e–mail: pawlowski@tpi.uni-jena.de

1 Introduction

Chiral gauge theories suffer after fermionic quantization from an anomalous breaking of gauge invariance due to the chiral anomaly [1]. The chiral anomaly is directly related (via cohomological descent equations [1]) to the anomalous Schwinger term in the algebra of the Gauss law operator G in these theories [2]. The Gauss law operator consists of two parts $G = G_A + G_\psi$, where G_A generates (time-independent) gauge transformations on the gauge field and $G_\psi = J^0$ acts on fermions. Here J^0 is the zero component of the (consistent) fermionic current. The algebra of G has been studied in many ways [2]-[14], since it is connected to the question of consistency of quantized chiral gauge theories. Whereas the cohomological prediction has been verified by the results, the algebras of $G_{A/\psi}$ do not coincide in general. Moreover, it is well known that an iterative calculation of double commutators containing $G_{A/\psi}$ leads to a violation of the Jacobi identity. Double commutators of fermionic currents obtained within an iterative computation do not fulfill a consistency condition, which relates them to the anomaly [10]. If one quantizes the gauge field, the gauge field part G_A is formally given by the covariant derivative of the chromo-electric field E^a ([5],[9],[14]). In these calculations an anomalous Schwinger term occurs in the commutator $[E^a, E^b]$. The double commutator $\left[E^a, [E^b, E^c]\right]$ obtained in an iterative computation violates the Jacobi identity. Therefore it is not clear whether the identification $G_A = -D \cdot E$ is correct. An explanation of these facts would give some insight to the structure of chiral gauge theories.

Since a violation of the Jacobi identity also takes place in the case of free fermionic axial and vector currents, one can study the free theory as a toy model. The Schwinger terms in the algebra of free currents [17],[18] have been shown to be operator valued. Considerations concerning the commutator algebra of composite operators in chiral gauge theories [19] indicate that this is also true for the Schwinger terms in the algebra of $G_{A/\psi}$.

In [10] a BJL-type prescription [22] was presented for the calculation of double commutators, which respects the Jacobi identity and fulfills the consistency condition mentioned before. Nevertheless it was not clear whether this prescription properly defines a double commutator, since in the free theory the prescription fails to give the correct result [18].

In the present paper we give an algebraic BJL-type derivation of commutators and double commutators of $G_{A/\psi}$ based only on consistency. This is related to the definition of a suitable regularization and renormalization of the time evolution operator U of the theory, and shows up in a non-trivial renormalization factor of U (e.g. [3],[4]). The derivation gives a proper definition of commutators and double commutators in chiral gauge theories. Since little is known about a consistent quantization of the gauge field in anomalous gauge theories, we treat the gauge field as an external field with $A(t = -\infty) = 0$. This defines an adiabatic solution of the theory [14]. We show that the Schwinger terms of commutators can be derived algebraically from the anomaly without further assumptions and without calculating diagrams.The present approach also provides a simple explanation for the discrepancies in the algebra of $G_{A/\psi}$ in the literature. The same method is applied to the calculation of double commutators. As in the case of commutators we can derive all Schwinger terms from the anomaly only. The Jacobi identity is fulfilled non-trivially for

the algebra of $G_{A/\psi}$. The result confirms the validity of the BJL-type prescription given in [10] for the double commutators of $G_{A/\psi}$. It also indicates, that this scheme should be valid in general.

In the second section we discuss the properties of the BJL-limit and give a definition of the Gauss law operator G at arbitrary times as the time evolution of $G(t=-\infty)$. The projective representation of the gauge group (on the Fock space) shows up in the properties of the time evolution operator. The following two sections are dedicated to the derivation of equal-time commutators and double commutators only using the anomaly equation. The last section summarizes the results.

2 The Gauss Law Operator

In the following the gauge field is treated as an external field. The fermionic action of a chiral gauge theory is

$$S[\bar\psi,\psi,A] \;=\; i\int \mathrm{d}^4x\bar\psi\left(\partial\!\!\!/+\frac{1-\gamma_5}{2}A\!\!\!/\right)\psi, \qquad j_a^\mu \;= i\bar\psi\frac{1-\gamma_5}{2}\gamma^\mu t^a\psi \tag{2.1}$$

with the notation

$$\begin{aligned} A &= A^a t^a, & \operatorname{Tr} t^a t^b &= -\frac{1}{2}\delta^{ab}, & [t^a,t^b] &= f^{abc}t^c \\ \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3, & \epsilon^{0123} &= \epsilon^{123}=1. \end{aligned} \tag{2.2}$$

The Gauss law operator G in gauge theories is the generator of time-independent gauge transformations and thus the generator of gauge transformations on the Fock space. The appropriate gauge in this framework is the Weyl gauge $A^0=0$. However, we can not neglect terms dependent on A^0 at intermediate steps, since $\frac{\delta}{\delta A^0}A^0$ contributes. Commutators of G (defined on the Fock space) are given by the equal-time commutators of

$$G^a(x) \;=\; -i\int \mathrm{d}x_0 D_i^{ab}(x)\frac{\delta}{\delta A_i^b(x)}+j_a^0(x), \qquad D_\mu^{ab} \;=\; \partial_\mu\delta^{ab}+f^{acb}A_\mu^c, \tag{2.3}$$

where the first part of G generates time independent gauge transformations on the (external) gauge field and the second part acts on fermions. However, for an anomalous gauge theory we have to be careful with this identification. Therefore we start at $t=-\infty$ with vanishing gauge field $A=0$, where the identification is justified. The time evolution of $G(t=-\infty,\boldsymbol{x})$ defines G at later times. Since we work with an external gauge field, the time evolution operator U is given by

$$\begin{aligned} U(-\infty,x_0) &\simeq \mathrm{T}^*\exp\left\{i\int \mathrm{d}^3x\int_{-\infty}^{x_0}\mathrm{d}tA\cdot j(x)\right\}, \\ U(y_0,x_0) &= U(-\infty,x_0)U(y_0,-\infty), \end{aligned} \tag{2.4}$$

where T^* is the Lorentz covariantized time ordered product. U has to be regularized and renormalized (e.g. [4] and references therein). If the representation of the gauge group is projective (on the Fock space), $U(x_0, x_0) = 1\!\!1$ can not be maintained in general. For our purpose it is sufficient to discuss the properties of

$$iW[A] \;=\; \ln\langle 0|\mathrm{T}^* \exp\left\{i\int \mathrm{d}^4x A\cdot j(x)\right\}|0\rangle \;=\; \ln\langle 0|U(-\infty,\infty)|0\rangle \tag{2.5}$$

Integrability (consistency) of $W[A]$ is crucial for the following calculations. Using a gauge covariant regularization of the current j [16], the integrability of $W[A]$ shows up in the (consistent) anomaly equation

$$\mathcal{A}^a(x) \;=\; D^{ac}_\mu(x)\frac{\delta}{\delta A^c_\mu(x)}W[A] \;=\; \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu \mathrm{Tr}\left[t^a\left(A_\nu\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma\right)\right]. \tag{2.6}$$

Eq. (2.6) also indicates the existence of a non-trivial renormalization factor in eq. (2.4). In the following we will only use the definition of $W[A]$ with a regularized current in eq. (2.5) and the anomaly eq. (2.6) as an input to calculate commutators and double commutators.

At $t = -\infty$ we deal with a free theory. $G(-\infty, \boldsymbol{x})$ (see eq. (2.3)) is given by

$$G(-\infty,\boldsymbol{x}) \;=\; G_A(-\infty,\boldsymbol{x}) + G_\psi(-\infty,\boldsymbol{x}), \tag{2.7}$$

where the fermionic part is the zero component of the consistent current J

$$G_\psi(-\infty,\boldsymbol{x}) \;=\; J^0(-\infty,\boldsymbol{x}) \quad \text{with} \quad J(-\infty,\boldsymbol{x}) \;=\; \frac{\delta}{\delta A(-\infty,\boldsymbol{x})}\int A\cdot j. \tag{2.8}$$

Since j in eq. (2.5) is gauge field dependent due to the regularization, J differs from j for arbitrary time by a term proportional to $\int A_\mu \frac{\delta}{\delta A} j^\mu$. G_A is the generator of time independent gauge transformations on the gauge field and is given by

$$G_A(-\infty,\boldsymbol{x}) \;=\; -i\int \mathrm{d}t D^{ab}_i(t,\boldsymbol{x})\frac{\delta}{\delta A^b_i(t,\boldsymbol{x})}. \tag{2.9}$$

The equal-time commutators of $G_{A/\psi}(-\infty,\boldsymbol{x})$ are the canonical commutators, since the gauge field vanishes ($A(-\infty,\boldsymbol{x}) = 0$). Now we define the Gauss law operator $G(x)$ for arbitrary times x_0 as the time evolution of $G(-\infty,\boldsymbol{x})$. The fermionic part $G_\psi(x)$ of G is given by the zero component of the consistent current J.

$$G_\psi(x) \;=\; J^0(x) \quad \text{with} \quad J(x) \;=\; U(x_0,-\infty)J(-\infty,\boldsymbol{x})U(-\infty,x_0). \tag{2.10}$$

The time evolution of G_A is given by

$$G_A(x) \;=\; U(x_0,-\infty)G_A(-\infty,\boldsymbol{x})U(-\infty,x_0) \;=\; -i\int \mathrm{d}t D^{ab}_i(t,\boldsymbol{x})\boldsymbol{\delta}^i_b(t,x), \tag{2.11}$$

where $\boldsymbol{\delta}$ is given by

$$
\begin{aligned}
\boldsymbol{\delta}_b^i(t,x) &= U(x_0,-\infty)U(-\infty,x_0)\frac{\delta}{\delta A_i^b(t,\boldsymbol{x})} + \Gamma_b^i(t,x) \\
\text{with} \quad \Gamma_b^i(t,x) &= i\theta(x_0-t)J_b^i(t,\boldsymbol{x}).
\end{aligned}
\tag{2.12}
$$

Γ^i can be seen as a non-trivial connection for the derivative $\frac{\delta}{\delta A}$ as in the two dimensional theory [14]. However, in contrast to [14] it is not possible to construct Γ^i explicitly in four dimensions. We relate expectation values of G_A to derivatives of $W[A]$ with respect to the gauge field. The expectation value of G_A is connected to $W[A]$ via

$$
\langle G_A(x)\rangle = \int \mathrm{d}t\ \theta(x_0-t)D_i^{ab}(t,\boldsymbol{x})\frac{\delta}{\delta A_i^b(t,\boldsymbol{x})}W[A], \tag{2.13}
$$

where the expectation value $\langle\cdots\rangle$ refers to the background defined by $W[A]$.

In the following we will present equal-time commutators and double commutators as derivatives of $W[A]$ with respect to the gauge field. The components $G_{A/\psi}$ of G are connected to the following A-derivatives

$$
\begin{aligned}
\tilde{G}_A^a(x) &= \int \mathrm{d}t\ \theta(x_0-t)D_i^{ab}(t,\boldsymbol{x})\frac{\delta}{\delta A_i^b(t,\boldsymbol{x})}, \\
\tilde{G}_\psi^a(x) &= \int \mathrm{d}t\ \theta(x_0-t)\partial_0\frac{\delta}{\delta A_0^a(t,\boldsymbol{x})} = \frac{\delta}{\delta A_0^a(x)}.
\end{aligned}
\tag{2.14}
$$

With eqs. (2.13),(2.14) we derive the well-known relation [21] between the time derivative of $G = G_A + G_\psi$ and the anomaly (using $A^0 = 0$)

$$
\partial_{x_0}\langle G^a(x_0,\boldsymbol{x})\rangle = \partial_{x_0}\tilde{G}(x)W[A] = D_\mu^{ab}\frac{\delta}{\delta A_\mu^b}W[A] = \mathcal{A}^a. \tag{2.15}
$$

The equal-time commutator of $G_A(x_0,\boldsymbol{x})$ with an operator

$$
\mathcal{O}(y) = U(y_0,-\infty)\mathcal{O}(-\infty,\boldsymbol{y})U(-\infty,y_0) \quad \text{with} \quad [A,\mathcal{O}] = 0 \tag{2.16}
$$

is given by

$$
\Big[U(x_0,-\infty)G_A(-\infty,\boldsymbol{x})U(-\infty,x_0), U(x_0,-\infty)\mathcal{O}(-\infty,\boldsymbol{y})U(-\infty,x_0)\Big]. \tag{2.17}
$$

With $D^i\frac{\delta}{\delta A^i}U(x_0,x_0) = 0$ we would conclude

$$
\Big[G_A(x_0,\boldsymbol{x}),\mathcal{O}(x_0,\boldsymbol{y})\Big] = U(x_0,-\infty)\Big[G_A(-\infty,\boldsymbol{y}),\mathcal{O}(-\infty,\boldsymbol{y})\Big]U(-\infty,x_0). \tag{2.18}
$$

Since we expect a projective representation of the gauge group, $D^i \frac{\delta}{\delta A^i} U(x_0, x_0) = 0$ can not be assumed. Indeed we have $\tilde{G}_A(-\infty, \boldsymbol{x}) U(x_0, x_0) \neq 0$. This can be taken into account by carefully calculating the equal-time limit

$$\begin{aligned} \lim_{t \to x_0} G_A(-\infty, \boldsymbol{x}) U(-\infty, t) U(x_0, -\infty) &= \lim_{t \to x_0} G_A(-\infty, \boldsymbol{x}) U(x_0, t) \\ = \; -i \lim_{p_0 \to \infty} \int \mathrm{d}t e^{ip_0(t - x_0} D_i^{ab}(t, \boldsymbol{x}) \frac{\delta}{\delta A_i^b(t, \boldsymbol{x})} U(-\infty, \infty) & \end{aligned} \tag{2.19}$$

The limit $p_0 \to \infty$ in the last line projects on the terms with $t = x_0$ in $D \frac{\delta}{\delta A} U(-\infty, \infty)$. Thus we conclude

$$\left\langle \left[G_A(x), \mathcal{O}(y) \right]_{ET} \right\rangle = -i \lim_{p_0 \to \infty} \int \mathrm{d}x_0 e^{ip_0(x_0 - y_0)} D_i^{ab}(x) \frac{\delta}{\delta A_i^b(x)} \left\langle \mathcal{O}(y) \right\rangle . \tag{2.20}$$

For operators $\mathcal{O}$ containing $\frac{\delta}{\delta A}$ we have additional terms proportional to $\frac{\delta}{\delta A}[G_A]$. Eq. (2.20) has the form of a BJL-limit [22], which connects the time-ordered product of two (bosonic) operators A, B with their equal time commutator. We have formally

$$\begin{aligned} \int \mathrm{d}x_0 e^{ip_0 x_0} \partial_{x_0} \mathrm{T} A(x) B(0) &= \int \mathrm{d}x_0 e^{ip_0 x_0} \partial_{x_0} \left[\theta(x_0) A(x) B(0) + \theta(-x_0) B(0) A(x) \right] \\ &= [A(x), B(0)]_{x_0 = 0} + \int \mathrm{d}x_0 e^{ip_0 x_0} \mathrm{T} \partial_{x_0} A(x) B(0). \end{aligned} \tag{2.21}$$

Providing a suitable regularization for the operators A and B, the second term vanishes in the limit $p_0 \to \infty$. The extension to double commutators is obvious [23].

$$\left[A(x), [B(y), C(0)] \right]_{ET} = \lim_{p_0 \to \infty} \lim_{q_0 \to \infty} \int \mathrm{d}x_0 \mathrm{d}y_0 e^{ip_0 x_0} e^{iq_0 y_0} \partial_{x_0} \partial_{y_0} \mathrm{T} A(x) B(y) C(0) \tag{2.22}$$

where the subscript ET denotes equal-time. In eq. (2.22) we have to perform first the $q_0 \to \infty$ limit. Moreover this only provides a proper definition of the double commutator, if the regularization of the time ordered product $\mathrm{T} A(x) B(y) C(0)$ does not affect the θ-functions. However, performing the BJL-limit perturbatively in Feynman integrals, this condition is violated by the exchange of integration and limit procedure. The diagrammatic calculation of double commutator with formally equivalent BJL-limits gives not the same result in general (e.g. fermionic currents [10],[13],[23]), which indicates the failure of the iterative BJL-limit. This is the reason for the violation of the Jacobi identity within the iterative BJL-procedure [13],[18].

Commutators containing G_A can be expressed as the limit of time-ordered products with the BJL-method. With use of eqs. (2.11),(2.20) we get ($\mathcal{O}$ does not contain $\frac{\delta}{\delta A}$)

$$\begin{aligned} \left\langle \left[G_A(x), \mathcal{O}(0) \right] \right\rangle &= \lim_{p_0 \to \infty} \int \mathrm{d}x_0 e^{ip_0 x_0} \partial_{x_0} \left\langle \mathrm{T}^* G_A(x) \mathcal{O}(0) \right\rangle \\ &= -i \lim_{p_0 \to \infty} \mathrm{d}t e^{ip_0 t} D_i^{ab}(t, \boldsymbol{x}) \frac{\delta}{\delta A_i^b(t, \boldsymbol{x})} \langle \mathcal{O}(0) \rangle, \end{aligned} \tag{2.23}$$

The covariantized time-ordering T^* appears naturally in the definition of the BJL-limit, if G_A is involved. However, if

$$\langle 0|\mathrm{T}^*\mathcal{O}(y)e^{i\int A\cdot j}|0\rangle \quad = \quad \langle 0|\mathrm{T}\mathcal{O}(y)e^{i\int A\cdot j}|0\rangle, \tag{2.24}$$

the results do not depend on the use of the usual time-ordering T or T*. The operators $\mathcal{O}$ mentioned here have the property eq. (2.24).

Eq. (2.23) can be extended to arbitrary commutators and double commutators of G_A, if we take into account derivatives of G_A with respect to the gauge field.

We want to emphasize that eq. (2.11) and the BJL-limit eq. (2.23) coincide with the BJL formulae for $-D\cdot E$ in a chiral theory with quantized gauge field, where only fermionic loops are taken into account [6].

3 The Algebra of Components of G

In the following we use the properties of $G_{A/\psi}, W[A]$ and the BJL-method for the calculation of the equal-time commutators of $G_{A/\psi}$. The results are derived only from the consistent anomaly. It is well known, that the Schwinger terms of the different commutators are related by functional derivatives of the anomaly (e.g. [7]). Given these relations we only have to calculate one Schwinger term from the (consistent) anomaly $\mathcal{A}$. First we derive the relations between Schwinger terms within the formalism introduced in section 2. It follows by eq. (2.23)

$$\begin{aligned}\left\langle\left[G^a(x), G^b_\psi(y)\right]_{ET}\right\rangle &= -i\lim_{p_0\to\infty}\int \mathrm{d}x_0 e^{ip_0(x_0-y_0)}\partial_0^x\tilde{G}^a(x)\left\langle J_b^0(y)\right\rangle \\ &= -i\lim_{p_0\to\infty}\int \mathrm{d}x_0 e^{ip_0(x_0-y_0)}\frac{\delta}{\delta A_0^b(y)}\partial_0^x\tilde{G}^a_\psi(x)W[A],\end{aligned} \tag{3.1}$$

where the integrability of $W[A]$ was used by commuting the derivatives with respect to A. With eq. (2.15) we conclude

$$\begin{aligned}\left\langle\left[G^a(x), G^b_\psi(y)\right]_{ET}\right\rangle &= -i\lim_{p_0\to\infty}\int \mathrm{d}x_0 e^{ip_0(x_0-y_0)}\frac{\delta}{\delta A_0^b(y)}\mathcal{A}^a(x) \\ &\quad +i\lim_{p_0\to\infty}\int \mathrm{d}x_0 e^{ip_0(x_0-y_0)}\frac{\delta}{\delta A_0^b(y)}f^{acd}A_0^c(x)\frac{\delta}{\delta A_0^d(x)}W[A] \\ &= \left\langle iG^{[a,b]}(x)\right\rangle\delta(\boldsymbol{x}-\boldsymbol{y}) - i\int \mathrm{d}x_0\frac{\delta}{\delta A_0^b(y)}\mathcal{A}^a(x).\end{aligned} \tag{3.2}$$

The relations between the other commutators follow similarly. We quote the results

$$
\begin{aligned}
\left\langle \left[G^a(x), G^b_A(y) \right]_{ET} \right\rangle &= i \left\langle G^{[a,b]}_A(x) \right\rangle \delta(\boldsymbol{x}-\boldsymbol{y}) + i \int \mathrm{d}x_0 D^{bd}_j(y) \frac{\delta}{\delta \partial_0 A^d_j(y)} \mathcal{A}^a(x) \\
\left\langle \left[G^a(x), G^b_\psi(y) \right]_{ET} \right\rangle &= i \left\langle G^{[a,b]}_\psi(x) \right\rangle \delta(\boldsymbol{x}-\boldsymbol{y}) - i \int \mathrm{d}x_0 \frac{\delta}{\delta A^b_0(y)} \mathcal{A}^a(x) \\
\left\langle \left[G^a(x), G^b(y) \right]_{ET} \right\rangle &= i \left\langle G^{[a,b]}(x) \right\rangle \delta(\boldsymbol{x}-\boldsymbol{y}) \\
&\quad - i \int \mathrm{d}x_0 \left[\frac{\delta}{\delta A^b_0(y)} - D^{bc}_i(y) \frac{\delta}{\delta \partial_0 A^c_i(y)} \right] \mathcal{A}^a(x).
\end{aligned}
\tag{3.3}
$$

Now we calculate the commutator $[G^a_A, G^b_A]$ with eq. (3.3) and considerations concerning symmetry properties. The anomalous Schwinger terms are connected to terms in $W[A]$ containing at least cubic powers of the gauge field. Hence the Schwinger terms contain at least linear powers of the gauge field. The only term with the correct symmetry properties is

$$
\left\langle \left[G^a_A(x), G^b_A(y) \right]_{ST} \right\rangle = q \epsilon^{ijk} D^{ac}_i(x) D^{bd}_j(y) \mathrm{Tr} \left[\{t^c, t^d\} A_k \right] \delta(\boldsymbol{x}-\boldsymbol{y}),
\tag{3.4}
$$

which is connected to a $U(1)$-curvature in field space [5],[9], [14]. In the present approach it follows by the observation, that the Schwinger term eq. (3.4) is directly related to $D^i \frac{\delta}{\delta A^i} U(x_0, x_0) \neq 0$, where $U(x_0, x_0)$ defines a loop in the gauge group. It remains to determine the constant q. Eq. (3.3) establishes the relation between the Schwinger term eq. (3.4) and $[G^a_\psi, G^a_A]_{ET}$

$$
\begin{aligned}
\left\langle \left[G^a_A(x), G^b_A(y) \right]_{ST} \right\rangle &= - \left\langle \left[G^a_\psi(x), G^b_A(y) \right]_{ET} \right\rangle \\
&\quad - \frac{i}{24\pi^2} \epsilon^{ijk} D^{bd}_j(y) \mathrm{Tr} \left[\{t^c, t^d\} \partial_i A_k(x) \right] \delta(x-y).
\end{aligned}
\tag{3.5}
$$

It follows from eqs. (3.4),(3.5) that

$$
\left\langle \left[G^a_\psi(x), G^b_A(y) \right]_{ET} \right\rangle \sim -q D^{bd}_j(y) \mathrm{Tr} \left[\{t^a, t^d\} A_k(x) \right] \partial^x_i \delta(\boldsymbol{x}-\boldsymbol{y}).
\tag{3.6}
$$

Thus we only have to evaluate these terms contributing to $[G^a_\psi, G^a_A]_{ST}$ to determine q. For this purpose we introduce the covariant current $\tilde{J}$, which differs from the consistent current by the Bardeen-Zumino polynomial ΔJ [15].

$$
\langle J^\mu_a \rangle = \left\langle \tilde{J}^\mu_a \right\rangle - \Delta J^\mu_a \quad \text{with} \quad \Delta J^\mu_a = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \left[t^a \{A_\nu, \partial_\rho A_\sigma\} + \frac{3}{2} A_\nu A_\rho A_\sigma \right].
\tag{3.7}
$$

We use the gauge covariance of $\langle \tilde{J} \rangle$ in the following derivation. With eq. (3.7) we have

$$
\begin{aligned}
\left\langle \left[G^a_\psi(x), G^b_A(y) \right]_{ET} \right\rangle &= i \lim_{p_0 \to \infty} \int \mathrm{d}y_0 e^{ip_0(y_0 - x_0)} D^{bd}_j(y) \frac{\delta}{\delta A^d_j(y)} \left\langle J^0_a(x) \right\rangle \\
&= i \lim_{p_0 \to \infty} \int \mathrm{d}y_0 e^{ip_0(y_0 - x_0)} D^{bd}_j(y) \frac{\delta}{\delta A^d_j(y)} \left[\left\langle \tilde{J}^0_a(x) \right\rangle - \Delta J^0_a(x) \right].
\end{aligned}
\tag{3.8}
$$

It follows from the covariance of $\langle\tilde{J}\rangle$ that the first term on the right hand side does not contribute to eq. (3.6). Thus, only taking into account terms which can contribute to eq. (3.6), we get ($\epsilon^{ijk}=\epsilon^{0ijk}$)

$$\begin{aligned}\left\langle\left[G_\psi^a(x),G_A^b(y)\right]_{ET}\right\rangle &\sim -i\lim_{p_0\to\infty}\int \mathrm{d}y_0 e^{ip_0(y_0-x_0)}D_j^{bd}(y)\frac{\delta}{\delta A_j^d(y)}\Delta J_a^0(x)\\ &= -\frac{i}{24\pi^2}\epsilon^{ijk}D_j^{bd}\mathrm{Tr}\left[\{t^c,t^d\}A_k(x)\right]\partial_i^x\delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned}\tag{3.9}$$

This determines $q=\frac{i}{24\pi^2}$ and we have finally

$$\begin{aligned}\left\langle\left[G_A^a(x),G_A^b(y)\right]_{ET}\right\rangle &= i\left\langle G_A^{[a,b]}(x)\right\rangle\delta(\boldsymbol{x}-\boldsymbol{y})\\ &\quad+\frac{i}{24\pi^2}\epsilon^{ijk}D_i^{ac}(x)D_j^{bd}(y)\mathrm{Tr}\Big[\{t^c,t^d\}A_k\Big]\delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned}\tag{3.10}$$

Together with eq. (3.3) this determines all commutators. The results coincide with the literature (e.g. [5],[7]).

4 Double Commutators

In the derivation of eq. (3.10) we used relations only valid as expectation values (see eqs. (3.5)-(3.10)). Thus we expect, that it is not possible to calculate the double commutator $\Big[G_A^a,[G_A^b,G_A^c]\Big]$ iteratively. Using the form of G_A (see eq. (2.11)) we conclude

$$\begin{aligned}&\left\langle\left[G_A^a(x),\left[G_A^b(y),G_A^c(z)\right]\right]_{ET}\right\rangle\\ =\;&\left\langle\left[G_A^a(x),iG_A^{[b,c]}(y)\right]_{ET}\right\rangle\delta(\boldsymbol{y}-\boldsymbol{z})\\ &+i\int_{t,t',t''}\Bigg\{\left[D_i^{ad}(t,\boldsymbol{x})\frac{\delta}{\delta A_i^d(t,\boldsymbol{x})}\left(D_j^{be}(t',\boldsymbol{y})D_k^{cf}(t'',\boldsymbol{z})\right)\right]\left\langle\left[\boldsymbol{\delta}_e^j(t',y),\boldsymbol{\delta}_f^k(t'',z)\right]\right\rangle\\ &+D_i^{ad}(t,x)D_j^{be}(t',y)D_k^{cf}(t'',z)\left\langle\left[\boldsymbol{\delta}_d^i(t,x),\left[\boldsymbol{\delta}_e^j(t',y),\boldsymbol{\delta}_f^k(t'',z)\right]\right]\right\rangle\Bigg\}.\end{aligned}\tag{4.1}$$

The first two terms follow easily with eq. (3.10), since they only involve derivatives of $\tilde{G}_A$ with respect to the gauge field. However, the last term can not be calculated with the known commutators. It follows from algebraic considerations, that it has to vanish, if the Jacobi identity is fulfilled. The structure of the double commutator

$$K_{def}^{ijk}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = i\int_{t,t',t''}D_i^{ad}(t,x)D_j^{be}(t',y)D_k^{cf}(t'',z)\left\langle\left[\boldsymbol{\delta}_d^i(t,x),\left[\boldsymbol{\delta}_e^j(t',y),\boldsymbol{\delta}_f^k(t'',z)\right]\right]\right\rangle\tag{4.2}$$

follows by dimensional analysis as

$$K_{def}^{ijk}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = q\epsilon^{ijk}D_i^{ad}(x)D_j^{be}(y)D_k^{cf}(z)\mathrm{Tr}\left[t^d\{t^e,t^f\}\right]\delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}).\tag{4.3}$$

The iterative result obtained with eq. (3.10) is $q = \frac{1}{24\pi^2}$. The Jacobi identity is only fulfilled when $q = 0$. Performing the BJL-limit we get for the last term in eq. (4.1)

$$
\begin{aligned}
K^{ijk}_{def}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= -\lim_{p_0\to\infty}\lim_{q_0\to\infty}\int \mathrm{d}x_0\mathrm{d}y_0\mathrm{d}z_0 e^{ip_0x_0}e^{iq_0y_0}\theta(-z_0)D^{ad}_i(x)D^{be}_j(y)D^{cf}_k(z) \\
&\quad\cdot\left[\frac{\delta}{\delta A^d_i(x)}\frac{\delta}{\delta A^e_j(y)}\frac{\delta}{\delta A^f_k(z)}W[A]\right] \\
&= 0.
\end{aligned} \tag{4.4}
$$

The only terms of

$$
\frac{\delta}{\delta A^d_i(x)}\frac{\delta}{\delta A^e_j(y)}\frac{\delta}{\delta A^f_k(z)}W[A],
$$

which contribute to eq. (4.4) are proportional to $\partial_0\delta(z_0-y_0)\delta(z_0-x_0)$, $\partial_0\delta(y_0-z_0)\delta(z_0-y_0)$ and $\partial_0\delta(z_0-y_0)\delta(x_0-x_0)$. The group structure is similar to eq. (3.10). It follows with the integrability of $W[A]$ that eq. (4.4) is proportional to

$$
\mathrm{Tr}\left[t^d\{t^e,t^f\}\right]\Big(\partial_0\delta(z_0-y_0)\delta(z_0-x_0) + \text{cycl. perms. of } (x_0,y_0,z_0)\Big) \;=\; 0. \tag{4.5}
$$

In an iterative BJL-limit one would only take into account one of the terms proportional to $\partial_0\delta(z_0-y_0)\delta(z_0-x_0)$ and $\partial_0\delta(x_0-z_0)\delta(x_0-y_0)$. However, only the sum of these two terms add up to zero in the limit, which is the reason for the violation of the Jacobi identity within an iterative calculation (the term proportional to $\partial_0\delta(x_0-z_0)\delta(y_0-z_0)$ is suppressed with $p_0/(p_0+q_0)$ in the BJL-limit in eq. (4.4)). With eq. (4.4) we have finally

$$
\begin{aligned}
&\left\langle\left[G^a_A(x),\left[G^b_A(y),G^c_A(z)\right]\right]_{ET}\right\rangle \\
=\;&\left\langle\left[G^a_A(x),iG^{[b,c]}_A(y)\right]_{ET}\right\rangle\delta(\boldsymbol{y}-\boldsymbol{z}) \\
&+\frac{1}{24\pi^2}\epsilon^{jkl}\int \mathrm{d}t\; D^{ad}_i(x)\left[\frac{\delta}{\delta A^d_i(x)}\Big(D^{be}_j(y)D^{cf}_k(z)\Big)\right]\mathrm{Tr}[\{t^e,t^f\}A_l]\delta(\boldsymbol{y}-\boldsymbol{z}) \\
=\;&\left\langle\left[G^a_A(x),[G^b_A(y),G^c_A(z)]\right]_{it}\right\rangle \\
&-\frac{1}{24\pi^2}\epsilon^{ijk}D^{ad}_i(x)D^{be}_j(y)D^{cf}_k(z)\mathrm{Tr}[t^d\{t^e,t^f\}]\delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z})
\end{aligned} \tag{4.6}
$$

The double commutator with the subscript it is the iterative double commutator.

Now we proceed as in the case of the commutators. We use the anomaly and the double commutator eq. (4.6) to calculate the other double commutators contributing to the algebra of G. As an important first step we prove, that double commutators with the structure $\left[G^a_{A/\psi},[G^b_{A/\psi},G^c]\right]$ agree with the iterative results. We derive with use of the

notation eq. (2.14)

$$\begin{aligned}
&\left\langle\left[G^a_{A/\psi}(x),\left[G^b_\psi(y),G^c(0)\right]\right]_{ET}\right\rangle\\
&= -\lim_{p_0\to\infty}\lim_{q_0\to\infty}p_0q_0\int \mathrm{d}y_0\mathrm{d}x_0 e^{ip_0x_0}e^{iq_0y_0}\left\langle \mathrm{T}^* G^a_{A/\psi}(x)G^b_\psi(y)G^c(0)\right\rangle\\
&= i\lim_{p_0\to\infty}p_0\int \mathrm{d}x_0 e^{ip_0x_0}\tilde G^a_{A/\psi}(x)\lim_{q_0\to\infty}q_0\int \mathrm{d}y_0 e^{iq_0y_0}\left\langle \mathrm{T}^* G^b_\psi(y)G^c(0)\right\rangle\\
&= \lim_{p_0\to\infty}p_0\int \mathrm{d}x_0 e^{ip_0x_0}\tilde G^a_{A/\psi}(x)\lim_{q_0\to\infty}q_0\int \mathrm{d}y_0 e^{iq_0y_0}\tilde G^b_\psi(y)\tilde G^c(0)W[A].
\end{aligned} \tag{4.7}$$

Here we used, that derivatives of $G^a_{A/\psi}$ with respect to the gauge field vanish in the BJL-limit with at least $p_0/(p_0+q_0)\to 0$. Performing the limit $q_0\to\infty$ we get

$$\begin{aligned}
&\left\langle\left[G^a_{A/\psi}(x),\left[G^b_\psi(y),G^c(0)\right]\right]_{ET}\right\rangle\\
&= i\lim_{p_0\to\infty}\int \mathrm{d}x_0 e^{ip_0x_0}\partial_0^x\tilde G^a_{A/\psi}(x)\left\langle G_\psi^{[b,c]}(0)\right\rangle\delta(\boldsymbol{y})\\
&\quad -\lim_{p_0\to\infty}p_0\int \mathrm{d}x_0 e^{ip_0x_0}\tilde G^a_{A/\psi}(x)\left[\lim_{q_0\to\infty}q_0\int \mathrm{d}y_0 e^{iq_0y_0}\tilde G^b_\psi(y)\int \mathrm{d}t\theta(-t)\mathcal{A}^c(t,0)\right]\\
&= \left\langle[G^a_{A/\psi}(x),iG_\psi^{[b,c]}(0)]_{ET}\right\rangle\delta(\boldsymbol{y})+\int \mathrm{d}x_0\partial_0\tilde G^a_{A/\psi}(x)\int \mathrm{d}y_0\frac{\delta}{\delta A_0^b(y)}\mathcal{A}^c(0).
\end{aligned} \tag{4.8}$$

For $G^a_{A/\psi}=G^a_\psi$ the second term in the last line vanishes. For $G^a_{A/\psi}=G^a_A$ we have

$$\int \mathrm{d}x_0\partial_0\tilde G^a_A(x) = \int \mathrm{d}x_0 D_i^{ad}(x)\frac{\delta}{\delta A_i^d(x)}.$$

Thus eq. (4.8) is the iterative result. In the derivation we used

$$\tilde G^c(x)W[A] = \int \mathrm{d}t\ \theta(x_0-t)\left(\mathcal{A}^c(t,\boldsymbol{x})-f^{cde}A_0^d(t,\boldsymbol{x})\frac{\delta}{\delta A_0^e(t,\boldsymbol{x})}W[A]\right). \tag{4.9}$$

The Schwinger term in eq. (4.8) is simply given by functional derivatives of the anomaly in contrast to the Schwinger term of $\left[G^a_A,[G^b_A,G^c_A]\right]$. It is directly related to the 2-cocycle in the algebra of the Gauss law operator G.

Applying the derivation of eq. (4.8) to the double commutator $\left[G^a_{A/\psi},[G^b_A,G^c]\right]$, we get

$$\begin{aligned}
&\left\langle\left[G^a_{A/\psi}(x),\left[G^b_A(y),G^c(0)\right]\right]_{ET}\right\rangle\\
&= -\lim_{p_0\to\infty}\lim_{q_0\to\infty}p_0q_0\int \mathrm{d}y_0\mathrm{d}x_0 e^{ip_0x_0}e^{iq_0y_0}\left\langle \mathrm{T}^* G^a_{A/\psi}(x)G^b_A(y)G^c(0)\right\rangle\\
&= i\lim_{p_0\to\infty}p_0\int \mathrm{d}x_0 e^{ip_0x_0}\tilde G^a_{A/\psi}(x)\lim_{q_0\to\infty}q_0\int \mathrm{d}y_0 e^{iq_0y_0}\left\langle \mathrm{T}^* G^b_A(y)G^c(0)\right\rangle\\
&\quad -\int \mathrm{d}t\theta(-t)D_i^{[b,c]d}(t,0)\left[\frac{\delta}{\delta A_i^d(t,0)}\tilde G^a_{A/\psi}(x)\right]W[A]\delta(\boldsymbol{y})\\
&= \left\langle[G_{A/\psi}(x),iG_A^{[b,c]}(0)]_{ET}\right\rangle\delta(\boldsymbol{y})-\int \mathrm{d}x_0\partial_0\tilde G^a_{A/\psi}(x)\int \mathrm{d}y_0 D_j^{bd}(y)\frac{\delta}{\delta\partial_0 A_j^d(y)}\mathcal{A}^c(0).
\end{aligned} \tag{4.10}$$

It follows with eqs. (4.8),(4.10), that double commutators of the form $\left[G^a_{A/\psi}, [G^b_{A/\psi}, G^c]\right]$ agree with the iterative results. Therefore they can be calculated from the known commutators eqs. (3.3),(3.10). The algebra for arbitrary combinations of $G_{A/\psi}$ is determined with the double commutators eqs. (4.6),(4.8),(4.10) and the result for $\left[G^a_\psi, [G^b_A, G^c_A]\right]$. We calculate

$$\begin{aligned}\left\langle\left[G^a_\psi(x),\left[G^b_A(y),G^c_A(0)\right]\right]_{ET}\right\rangle &= \left\langle\left[G^a_\psi(x), iG^{[b,c]}_A(0)\right]_{ET}\right\rangle\delta(\boldsymbol{y}) \\ &\quad -\lim_{p_0\to\infty}\lim_{q_0\to\infty}\int \mathrm{d}x_0\mathrm{d}y_0\mathrm{d}z_0 e^{ip_0x_0}e^{iq_0y_0}\theta(-z_0) \\ &\quad \cdot D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}\partial_0\frac{\delta}{\delta A^a_0(x)}W[A].\end{aligned} \tag{4.11}$$

Using

$$\partial_0\frac{\delta}{\delta A^0_a(x)}W[A] = \mathcal{A}^a(x)-\left[D^{af}_k(x)\frac{\delta}{\delta A^f_k(x)}+f^{agf}A^g_0(x)\frac{\delta}{\delta A^f_0(x)}\right]W[A]$$

it follows

$$\begin{aligned}\left\langle\left[G^a_\psi(x),\left[G^b_A(y),G^c_A(0)\right]\right]_{ET}\right\rangle &= \left\langle\left[G^a_\psi(x), iG^{[b,c]}_A(0)\right]_{ET}\right\rangle\delta(\boldsymbol{y}) \\ &\quad -\int \mathrm{d}x_0\mathrm{d}y_0 D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta\partial_0 A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}\mathcal{A}^a(x) \\ &\quad +\int \mathrm{d}x_0\mathrm{d}y_0 D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta\partial_0 A^e_j(z)}\mathcal{A}^a(x) \\ &\quad +\lim_{p_0\to\infty}\lim_{q_0\to\infty}\int \mathrm{d}x_0\mathrm{d}y_0 dz_0 e^{ip_0x_0}e^{iq_0y_0}\theta(-z_0) \\ &\quad \cdot D^{bd}_i(y)D^{ce}_j(z)\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}D^{af}_k(x)\frac{\delta}{\delta A^f_k(x)}W[A].\end{aligned} \tag{4.12}$$

The terms containing functional derivatives of the anomaly have the form

$$\begin{aligned}&\int \mathrm{d}x_0\int \mathrm{d}y_0 D^{bd}_i(y)D^{ce}_j(z)\left[\frac{\delta}{\delta A^d_i(y)}\frac{\delta}{\delta\partial_0 A^e_j(z)}-\frac{\delta}{\delta\partial_0 A^d_i(y)}\frac{\delta}{\delta A^e_j(z)}\right]\mathcal{A}^a(x) \\ &= \frac{1}{24\pi^2}\epsilon^{ijk}D^{bd}_i(y)D^{ce}_j(z)\partial^x_k\mathrm{Tr}\left[t^a\{t^d,t^e\}\right]\delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{z}).\end{aligned} \tag{4.13}$$

The last term on the right hand side of eq. (4.12) consists of equal-time commutators of the form $[\boldsymbol{\delta}_{[a,g]},\boldsymbol{\delta}_h]$ (see eq. (3.10)) and terms proportional to $\delta^3_A W[A]$, which vanish in the

BJL-limit. Performing the BJL-limit, we are led to

$$\begin{aligned}
&\left\langle \left[G_\psi^a(x), \left[G_A^b(y), G_A^c(z) \right] \right]_{ET} \right\rangle \\
= \ & \left\langle \left[G_\psi^a(x), iG_A^{[b,c]}(y) \right]_{ET} \right\rangle \delta(\boldsymbol{y}-\boldsymbol{z}) + \frac{1}{24\pi^2}\epsilon^{ijk} D_i^{bd}(y) D_j^{ce}(z) \partial_k^x \\
& \cdot \mathrm{Tr}\left[t^a \{ t^d, t^e \} \right] \delta(\boldsymbol{x}-\boldsymbol{y}) \delta(\boldsymbol{x}-\boldsymbol{z}) - i \int \mathrm{d}t \mathrm{d}t' D_i^{bd}(t,\boldsymbol{y}) D_j^{ce}(t',\boldsymbol{z}) \\
& \cdot \Big\{ \left\langle \left[\boldsymbol{\delta}_{[a,d]}^i(t,y), \boldsymbol{\delta}_e^j(t',z) \right] \right\rangle \delta(\boldsymbol{x}-\boldsymbol{y}) - \left\langle \left[\boldsymbol{\delta}_{[a,e]}^j(t',z), \boldsymbol{\delta}_d^i(t,y) \right] \right\rangle \delta(\boldsymbol{x}-\boldsymbol{z}) \Big\} \\
= \ & \left\langle \left[G_\psi^a(x), iG_A^{[b,c]}(y) \right]_{ET} \right\rangle \delta(\boldsymbol{y}-\boldsymbol{z}) \\
& + \frac{1}{24\pi^2}\epsilon^{ijk} D_i^{bd}(y) D_j^{ce}(z) \partial_k^x \mathrm{Tr}\left[t^a \{ t^d, t^e \} \right] \delta(\boldsymbol{x}-\boldsymbol{y}) \delta(\boldsymbol{x}-\boldsymbol{z}) \\
& - \epsilon^{ijk} \frac{1}{24\pi^2} D_i^{bd}(y) D_j^{ce}(z) \mathrm{Tr}\Big[\Big(\{ [t^a, t^d], t^e \} + \{ [t^a, t^e], t^d \} \Big) A_k(x) \Big] \delta(\boldsymbol{x}-\boldsymbol{y}) \delta(\boldsymbol{x}-\boldsymbol{z}) \\
= \ & \left\langle \left[G_\psi^a(x), iG_A^{[b,c]}(y) \right]_{ET} \right\rangle \delta(\boldsymbol{y}-\boldsymbol{z}) \\
& + \frac{1}{24\pi^2}\epsilon^{ijk} D_i^{bd}(y) D_j^{ce}(z) D_k^{af}(x) \mathrm{Tr}\left[t^f \{ t^d, t^e \} \right] \delta(\boldsymbol{x}-\boldsymbol{y}) \delta(\boldsymbol{x}-\boldsymbol{z}).
\end{aligned} \tag{4.14}$$

The non-iterative part in eq. (4.14) is exactly the same as for the double commutator eq. (4.6) with a relative minus sign. We conclude

$$\left\langle \left[G^a(x), \left[G_A^b(y), G_A^c(z) \right] \right]_{ET} \right\rangle = \left\langle \left[G^a(x), \left[G_A^b(y), G_A^c(z) \right] \right]_{it} \right\rangle . \tag{4.15}$$

We want to emphasize, that the non-iterative part of eq. (4.14) is also connected to the fact, that the result for the commutator $[G_A^a, G_A^b]$ (see eq. (3.10)) is not valid on the operator level. With eqs. (4.8),(4.10),(4.15) it follows, that double commutator containing at least one full G are given by the iterative results. We collect the results

$$\begin{aligned}
\left\langle \left[G_{A/\psi}^a(x), \left[G_{A/\psi}^b(y), G^c(z) \right] \right]_{ET} \right\rangle &= \left\langle \left[G_{A/\psi}^a(x), \left[G_{A/\psi}^b(y), G^c(z) \right] \right]_{it} \right\rangle \\
\left\langle \left[G^a(x), \left[G_{A/\psi}^b(y), G_{A/\psi}^c(z) \right] \right]_{ET} \right\rangle &= \left\langle \left[G^a(x), \left[G_{A/\psi}^b(y), G_{A/\psi}^c(z) \right] \right]_{it} \right\rangle .
\end{aligned} \tag{4.16}$$

Eq. (4.16), the commutators eq. (3.3),(3.10) and the double commutator eq. (4.6) fix all double commutators, as already mentioned. The algebra of two G_A and one G_ψ is given by

$$\begin{aligned}
\left\langle \left[G_A^a(x), \left[G_A^b(y), G_\psi^c(z) \right] \right]_{ET} \right\rangle &= \left\langle \left[G_A^a(x), \left[G_A^b(y), G_\psi^c(z) \right] \right]_{it} \right\rangle + \frac{1}{24\pi^2}\epsilon^{ijk} D_i^{ad}(x) \\
& \quad \cdot D_j^{be}(y) D_k^{cf}(z) \mathrm{Tr}\left[t^d \{ t^e, t^f \} \right] \delta(\boldsymbol{x}-\boldsymbol{y}) \delta(\boldsymbol{y}-\boldsymbol{z}) \\
\left\langle \left[G_\psi^a(x), \left[G_A^b(y), G_A^c(z) \right] \right]_{ET} \right\rangle &= \left\langle \left[G_\psi^a(x), iG_A^{[b,c]}(y) \right]_{it} \right\rangle + \frac{1}{24\pi^2}\epsilon^{ijk} D_i^{ad}(x) \\
& \quad \cdot D_j^{be}(y) D_k^{cf}(z) \mathrm{Tr}\left[t^d \{ t^e, t^f \} \right] \delta(\boldsymbol{x}-\boldsymbol{y}) \delta(\boldsymbol{y}-\boldsymbol{z})
\end{aligned} \tag{4.17}$$

The algebra of two G_ψ and one G_A is given by

$$
\begin{aligned}
\left\langle \left[G_\psi^a(x), \left[G_A^b(y), G_\psi^c(z)\right]\right]_{ET} \right\rangle &= -\frac{1}{24\pi^2}\epsilon^{ijk} D_i^{ad}(x) D_j^{be}(y) D_k^{cf}(z) \\
&\quad \cdot \mathrm{Tr}\left[t^d\{t^e, t^f\}\right] \delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}) \\
\left\langle \left[G_A^a(x), \left[G_\psi^b(y), G_\psi^c(z)\right]\right]_{ET} \right\rangle &= \left\langle \left[G_A^a(x), \left[G_\psi^b(y), G_\psi^c(z)\right]\right]_{it} \right\rangle \\
&\quad -\frac{1}{24\pi^2}\epsilon^{ijk} D_i^{ad}(x) D_j^{be}(y) D_k^{cf}(z) \mathrm{Tr}\left[t^d\{t^e, t^f\}\right] \\
&\quad \cdot \delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z})
\end{aligned}
\tag{4.18}
$$

The algebra of the fermionic currents is given by

$$
\begin{aligned}
\left\langle \left[G_\psi^a(x), \left[G_\psi^b(y), G_\psi^c(z)\right]\right]_{ET} \right\rangle &= \left\langle \left[G_\psi^a(x), iG_\psi^{[b,c]}(y)\right]_{ET} \right\rangle \delta(\boldsymbol{y}-\boldsymbol{z}) \\
&\quad +\frac{1}{24\pi^2}\epsilon^{ijk} D_i^{ad}(x) D_j^{be}(y) D_k^{cf}(z) \mathrm{Tr}\left[t^d\{t^e, t^f\}\right] \\
&\quad \cdot \delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z})
\end{aligned}
\tag{4.19}
$$

It can be shown explicitly in a long, but straightforward calculation, that the Jacobi identity is fulfilled for all these results. Since we have related the double commutators to derivatives of $W[A]$ with respect to the gauge field, this provides a convenient way to prove the Jacobi identity (integrability of $W[A]$).

5 Discussion

In this paper a purely algebraic BJL-type derivation of the algebra of the Gauss law operator G in chiral gauge theories was given. The theory has been defined in an external gauge field (adiabatic solution). The expectation values of commutators and double commutators can be expressed as derivatives of $W[A]$ with respect to the gauge field. All Schwinger terms follow with the anomaly without any further assumption.

The Schwinger terms depend on the properties of the time evolution operator U, which defines the underlying theory. If the representation of the gauge group is projective (on the Fock space), we have $D^i \frac{\delta}{\delta A^i} U(x_0, x_0) \neq 0$ (see section 2). This is the reason for the Schwinger term in the commutator $[G_A^a, G_A^b]$ (eq. (3.10)). We want to emphasize that the explanation of this Schwinger term is based purely on an adiabatic solution of the theory in an external gauge field. There is no need to introduce a quantized gauge field to obtain this result.

The commutators (eq. (3.3) and eq. (3.10)) coincide with the BJL-results obtained via the explicit calculation of diagrams [5], or path integral methods [7], where only fermionic loops give contributions. In contrast to these approaches we do not need to calculate diagrams nor do we need further assumptions as in [7]. The algebraic derivation in [7] was

based on the assumption, that $[G^a_A, G^b_A]_{ST} = 0$ for G defined with the covariant current. Adding the Bardeen-Zumino polynomial ΔJ^0 to G_ψ we get the results for G defined with the covariant current (e.g. [7],[8],[9]).

It is also possible to relate the Schwinger terms of $G_{A/\psi}$ obtained in the present paper to those in [11],[12]. Although the consistent current was used in [11],[12] and the algebra of G coincides with eq. (3.3), the Schwinger terms in the algebra of $G_{A/\psi}$ are different. This can be explained by studying the time evolution operator U. If we assume that

$$U(x_0, x_0) \quad = \quad 1\!\!1 \tag{5.1}$$

in eq. (2.18) we would arrive at the results of [11]. In [11] eq. (5.1) was used implicitely, since the commutators of $G_{A/\psi}$ were calculated as defined in eq. (2.18). Condition eq. (5.1) is equivalent to gauge invariance of the underlying theory. In [12] such a theory was constructed by adding a Wess-Zumino field. In this theory G is not the full Gauss law operator, since G does not generate gauge transformations on the Wess-Zumino field. The results in [11] coincide with those of [12], which confirms the interpretation.

The double commutators have been derived by using the integrability of $W[A]$. Double commutators, which contain at least one G agree with the iterative results. This indicates, that G is a well defined operator on quantum level. The Jacobi identity is satisfied by our results (eqs. (4.6),(4.17),(4.18) and (4.19). This is an outcome of the integrability (consistency) of $W[A]$, since the double commutators are related to derivatives of $W[A]$ with respect to the gauge field. In an iterative calculation one drops terms ensuring the integrability (see eq. (4.4) and the following discussion), which leads to the violation of the Jacobi identity. Since double commutators containing at least one G agree with the iterative result, there is only one non-iterative term (up to a minus sign)

$$\frac{1}{24\pi^2}\epsilon^{ijk} D^{ad}_i(x) D^{be}_j(y) D^{cf}_k(z) \mathrm{Tr}\left[t^d\{t^e, t^f\}\right] \delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}) \tag{5.2}$$

This can be traced back to the fact, that the gauge group representation is projective and is related directly to the Schwinger term in the commutator $\left[G^a_A, G^b_A\right]$.

The non-iterative terms of double commutators, which are related via cyclic permutation of the operators, are identical (no relative minus sign) (see eq. (4.17)-eq. (4.19)). The BJL-type prescription of [10] only works in this case. Therefore the results here confirm the validity of the prescription in the case of $G_{A/\psi}$. Moreover, the connection of the non-iterative term eq. (5.2) to the integrability of the generating functional $W[A]$ indicates that this should hold for arbitrary double commutators.

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